SOME RESULTS ON Q–INTEGRAL GRAPHS

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Abstract

We consider the problem of determining the Q–integral graphs, i.e. the graphs with integral signless Laplacian spectrum. First, we determine some infinite series of such graphs having the other two spectra (the usual one and the Laplacian) integral. We also completely determine all $(2, s)$–semiregular bipartite graphs with integral signless Laplacian spectrum. Finally, we give some results concerning $(3, 4)$ and $(3, 5)$–semiregular bipartite graphs with the same property.

1 Introduction

Let G be a graph with adjacency matrix A $(= A_G)$. The eigenvalues and the spectrum of A are also called the *eigenvalues* and the *spectrum* of G, respectively. A graph whose spectrum consists entirely of integers is called an *integral graph*. If we consider a matrix $L = D - A$ instead of A, where D is the diagonal matrix of vertex–degrees (in G), we get the *Laplacian eigenvalues* and the *Laplacian spectrum*, while in the case of matrix $Q = D + A$ we get the *signless Laplacian eigenvalues* and the *signless Laplacian spectrum*, respectively. For short, the signless Laplacian eigenvalues and the signless Laplacian spectrum will be called the Q*–eigenvalues* and the Q*–spectrum*, respectively. A graph whose Laplacian (resp. signless Laplacian)

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spectrum consists entirely of integers is called an *L–integral* (resp. *Q–integral*) graph. Also, we say that a graph is *ALQ–integral* if it has all three mentioned spectra integral.

Let R $(= R_G)$ be the $n \times m$ vertex–edge incidence matrix of G. Denote by $L(G)$ the *line graph* of G (recall, vertices of $L(G)$ are in one–to–one correspondence with edges of G, and two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are adjacent). The following relations are well known (see, for example, [2]):

$$RR^T = A_G + D, \quad R^T R = A_{L(G)} + 2I,$$

From these relations it immediately follows that

$$P_{L(G)}(\lambda) = (\lambda + 2)^{m-n} Q_G(\lambda + 2), \tag{1}$$

where $Q_G(\lambda) = \det(\lambda I - Q)$ is the characteristic polynomial of the matrix Q.

The integral and L–integral graphs are well studied in the literature. On the other hand, the graphs with integral Q–spectrum are studied in exactly two papers [9] and [10], so far. Since the matrix Q is positive semidefinite, the Q–spectrum consists of non–negative values. Furthermore, the least eigenvalue of the signless Laplacian of a connected graph is equal to 0 if and only if the graph is bipartite; in this case 0 is a simple eigenvalue (see [2], Proposition 2.1).

Recall that if G is a regular graph which is integral in the sense of any of spectra mentioned above, then it has integral the other two spectra (cf. [2], Section 3), as well. In particular, the complete graphs form an infinite series of graphs having all three spectra integral. Also, if G is a bipartite graph then its L–spectrum and Q–spectrum coincide (the proof can be found in many places, see [4], for example), and therefore every bipartite graph is L–integral if and only if it is Q–integral.

In Section 2, we mention some results from the literature in order to make the paper more self–contained. In Section 3, we identify some infinite series of ALQ–integral graphs. All $(2, s)$– semiregular bipartite Q–integral graphs are determined in Section 4. Some possible Q–spectra of connected Q–integral $(3, 4)$–semiregular bipartite graphs obtained in [9] are considered in Section 5. In addition, we give the possible Q–spectra of connected Q–integral $(3, 5)$–semiregular bipartite graphs and consider some of them.

2 Preliminaries

Recall that for an arbitrary edge of a graph G, the *edge–degree* is the number of edges adjacent to it. Also, we say that G is *edge–regular* if its edges have the same edge–degree. Further, an (r, s)*–semiregular bipartite* graph is a bipartite graph whose each vertex in the first (resp. second) colour class has degree r (resp. s).

Following [9] and [10], we list some results regarding Q–integral graphs. All Q–integral graphs with maximum edge–degree at most 4 are known; exactly 26 of them are connected. Also, all (r, s)–semiregular bipartite graphs with $r + s = 7$ and $r < 3 < s$ are known (note, an edge–regular graph with edge degree 5 is in fact an (r, s)–semiregular bipartite graph with $r + s = 7$); exactly 3 of them are connected. In addition, all possible Q–spectra of connected $(3, 4)$–semiregular bipartite graphs are determined, and all graphs having some of those Q–spectra are identified. The remaining unsolved cases are given in Table 1 (a complete table which contains all 16 possible spectra can be found in [9]): each row contains the number of vertices (n), the number of edges (m), the multiplicities of eigenvalues $0, 1, ..., 7$, the number of quadrangles (q) and the number of hexagons (h). Finally, all Q–integral graphs up to 10 vertices are known; exactly 172 of them are connected.

n	m	0	1	2	3	4	5	6	7	q	h
21	36	1	4	0	7	4	0	4	1	18	0
21	36	1	3	3	5	2	3	3	1	12	28
21	36	1	2	6	3	0	6	2	1	6	56
28	48	1	4	6	5	1	6	4	1	6	44
28	48	1	5	3	7	3	3	5	1	12	16
35	60	1	6	6	7	2	6	6	1	6	32
35	60	1	7	3	9	4	3	7	1	12	4
42	72	1	8	6	9	3	6	8	1	6	20
49	84	1	10	6	11	4	6	10	1	6	8

Table 1

Now we list some notions and results to be used later on (see [2], Section 4; especially Theorem 4.1 and Corollaries 4.2 and 4.3). A *semi–edge walk* (of length k) in a graph G is an alternating sequence $v_1, e_1, v_2, e_2, ..., v_k, e_k, v_{k+1}$ of vertices $v_1, v_2, ..., v_{k+1}$ and edges $e_1, e_2, ..., e_k$ such that for any $i = 1, 2, ..., k$ the vertices v_i and v_{i+1} are end–vertices (not necessarily distinct) of the edge e_i. Let Q be the signless Laplacian of a graph G. Then the (i, j)–entry of the matrix Q^k is equal to the number of semi–edge walks starting at vertex i and terminating at vertex j. Let $T_k = \sum_{i=1}^{n} \mu_i^k$ $(k = 0, 1, ...)$ be the k–th spectral moment for the Q–spectrum (here $\mu_1, \mu_2, \ldots, \mu_n$ are the Q–eigenvalues of G). Then T_k is equal to the number of closed semi–edge walks of length k. In particular, if G has n vertices, m edges, t triangles, and vertex–degrees $d_1, d_2, \ldots, d_n$, then

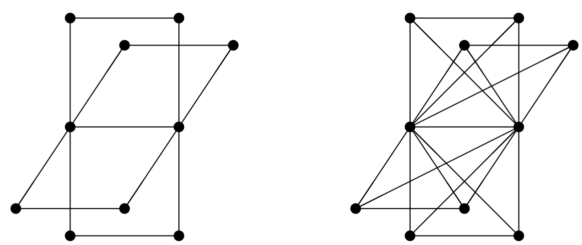
$$T_0 = n, \quad T_1 = \sum_{i=1}^{n} d_i = 2m, \quad T_2 = 2m + \sum_{i=1}^{n} d_i^2, \quad T_3 = 6t + 3\sum_{i=1}^{n} d_i^2 + \sum_{i=1}^{n} d_i^3.$$

Finally, if G is an (r,s)–semiregular bipartite graph which contains q quadrangles and h hexagons then for the spectral moments T_k $(k = 4, 5, 6)$ we have (cf. [9], Lemma 3.2)

$$\begin{aligned}
T_4 = &\ \Big(r^3 + s^3 + 4(r^2 + s^2) + 2(r + s) + 4rs - 2\Big)m + 8q,\\
T_5 = &\ \Big(r^4 + 5(r^3 + r^2 - r) + s^4 + 5(s^3 + s^2 - s) + 5rs(r + s + 2)\Big)m +\\
&\ 20(r + s)q,\\
T_6 = &\ \Big(r^5 + s^5 + 6(r^4 + s^4) + 9(r^3 + s^3) - 7(r^2 + s^2) - 6(r + s + rs) +\\
&\ 6rs(r^2 + s^2 + rs) + 21(r^2 s + s^2 r) + 4\Big)m + 12\Big(3(r^2 + s^2) +\\
&\ 2(r + s) + 4rs - 4\Big)q + 12h.
\end{aligned}$$

3 Some infinite series of ALQ–integral graphs

The problem of determining the connected non–regular graphs that are integral, Laplacian integral and signless Laplacian integral was set in [11] (see Problem C). The determination of infinite series of such graphs also merits attention. An example is a series of complete bipartite graphs $K_{m,n}$ such that mn is a perfect square (see [10], Lemma 2). Furthermore, following the same paper we learn that exactly 42 connected graphs up to 10 vertices have all three spectra integral, while 40 of them are either regular or complete bipartite or both. The remaining two graphs are $K_2 + K_{1,4}$ and $K_2 \nabla 4K_2$ (see Fig. 1). Here, $+$ stands for the sum of two graphs, while ∇ denotes the join of two graphs; recall, the join (or the complete product) of two graphs is the graph obtained by joining every vertex of the first graph with every vertex of the second graph. These are connected graphs of smallest order (which are neither regular nor complete bipartite) being integral in the sense of all three spectra, and we shall generalize this result.



$K_2 + K_{1,4}$ $\qquad$ $K_2 \nabla 4K_2$

Fig. 1

First we prove the following theorem.

Theorem 3.1 *If $G_1 + G_2$ is a bipartite graph, where both G_1 and G_2 are integral and L–integral then $G_1 + G_2$ is ALQ–integral.*

Proof If $\lambda_1^{(1)}, ..., \lambda_{n_1}^{(1)}$ and $\lambda_1^{(2)}, ..., \lambda_{n_2}^{(2)}$ are the eigenvalues (resp. the Laplacian eigenvalues) of G_1 and G_2 then the eigenvalues (resp. the Laplacian eigenvalues) of $G_1 + G_2$ are $\lambda_i^{(1)} \pm \lambda_j^{(2)}, (1 \leq i \leq n_1, 1 \leq j \leq n_2)$ (see [1], p. 70, and [8], p. 150). In addition, we have that for bipartite graphs Laplacian and signless Laplacian spectra coincide (see Section 1). Hence, the graph $G_1 + G_2$ has all three spectra integral.

This completes the proof. $\square$

Corollary 3.1 *The graph $K_{m_1,n_1} + K_{m_2,n_2}$ is ALQ–integral if both m_1n_1 and m_2n_2 are the perfect squares. Consequently, the graph $K_2 + K_{1,n}$ is ALQ–integral whenever n is a perfect square.*

Proof The proof follows from the previous theorem and the mentioned fact that a complete bipartite graph $K_{m,n}$ is *ALQ*–integral if and only if mn is a perfect square. $\square$

In what follows we construct another infinite series of *ALQ*–integral graphs.

Lemma 3.1 *The graph $K_2 \nabla nK_2$ is integral if and only if n is a perfect square. The same graph is L–integral for each n.*

Proof The lemma is obviously true for $n = 0$. Assume now, $n \geq 1$. By using the formula for the characteristic polynomial of a join of two graphs (compare [1], Theorem 2.8) we get that $K_2 \nabla nK_2$ has the following spectrum: $\{(-1)^{n+1},\ 1^{n-1},\ \pm 2\sqrt{n} + 1\}$. (In the exponential notation the exponents stand for the multiplicities of the eigenvalues.) Similarly, by using the formula for the Laplacian characteristic polynomial of a join of two graphs (see [1], p. 58) we get that $K_2 \nabla nK_2$ has the following Laplacian spectrum: $\{0,\ 2^{n-1},\ 4^n,\ (2n+2)^2\}$, and the proof follows. $\square$

Before we consider the Q–spectrum of $K_2 \nabla nK_2$ we prove the next (general) result. Recall that if G is an arbitrary (simple) graph and u its vertex then open and closed neighbourhoods of u are $\{v \mid v \sim u\}$ and $\{v \mid v \sim u\} \cup \{u\}$, respectively. We say that two vertices are *duplicate* (*coduplicate*) if their open (resp. closed) neighbourhoods are the same.

Lemma 3.2 *Any collection of k mutually duplicate (resp. coduplicate) vertices of degree d in a simple graph G gives $k-1$ Q–eigenvalues of G all equal to d (resp. $d-1$).*

Proof Any pair of duplicate (resp. coduplicate) vertices u, v gives rise to a signless Laplacian eigenvector of G for d (resp. $d-1$) defined as follows: all its entries are zero except those corresponding to u and v which can be taken to be 1 and -1, or vice versa. Thus any collection with k mutually duplicate (resp. coduplicate) vertices gives rise to $k-1$ linearly independent Q–eigenvectors for d (resp. $d-1$).

The proof is complete. $\square$

Lemma 3.3 *The Q–spectrum of a graph $K_2 \nabla nK_2$ $(n \geq 1)$ consists of the following eigenvalues: $2^{n+1}, 4^{n-1}, 2n$ and $2n+4$.*

Proof First, there are two coduplicate vertices of degree $2n+1$, and there are n pairs of mutually coduplicate vertices of degree 3. By the previous lemma, we deduce that the Q–spectrum of our graph contains the eigenvalue $2n$ as well as the eigenvalue 2 with the multiplicity at least n. The next n eigenvalues we get by constructing the corresponding eigenvectors. The matrix $Q = A + D$ has the form:

$$Q = \begin{pmatrix} 2n+1 & 1 & 1 & 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 2n+1 & 1 & 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & 3 & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 1 & 1 & 3 & 0 & 0 & \dots & 0 & 0 \\ 1 & 1 & 0 & 0 & 3 & 1 & \dots & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 3 & \dots & 0 & 0 \\ \vdots & & & & & & \ddots & & \vdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots & 3 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots & 1 & 3 \end{pmatrix}.$$

Now, it is a mater of routine to check that the eigenvector $x_1 = (n, n, 1, 1, \dots, 1)^T$ corresponds to eigenvalue $2n+4$, while the following $n-1$ linearly independent eigenvectors $x_2 = (0, 0, -1, -1, 1, 1, 0, 0, \dots, 0)^T$, $x_3 = (0, 0, -1, -1, -1, -1, 2, 2, 0, 0, \dots, 0)^T$, ... and $x_n = (0, 0, -1, -1, \dots, -1, -1, n-1, n-1)^T$ correspond to eigenvalue 4.

So far, we have $2n-1$ eigenvalues of Q. By summing them we get $10n$. On the other hand, the trace of Q is equal to $10n+2$, and therefore the remaining eigenvalue is equal to 2.

The proof is complete. □

Collecting the results above we get the following theorem.

Theorem 3.2 *The graph $K_2 \nabla nK_2$ is ALQ–integral if and only if n is a perfect square.*

4 Q–integral $(2, s)$–semiregular bipartite graphs

It is known that each complete bipartite graph is Q–integral (see [10], Lemma 1), and consequently, a $(2, s)$–semiregular complete bipartite graph is Q–integral for every non–negative integer s.

Denote by $S(G)$ the subdivision of a graph G; recall, the subdivision of a graph G is obtained by inserting into each of its edges a vertex of degree 2 (see also [1], p. 16). Now we prove the following lemma.

Lemma 4.1 *Each $(2, s)$–semiregular bipartite graph G is a subdivision of some s–regular multigraph*[1] *G'. In addition, G does not contain any quadrangle as an induced subgraph if and only if G' is a graph.*

Proof Take any $(2, s)$–semiregular bipartite graph G. If we replace each path of the length 2 between the vertices of degree s by a single edge we get the corresponding multigraph.

Now, if G does not contain any quadrangle as an induced subgraph then there are no two different paths of the length 2 between two vertices of degree s (in G). Thus, G' has no multiple edges. Finally, the subdivision of a regular graph of degree s is a $(2, s)$–semiregular bipartite graph having no vertices with the same open neighbourhood and so it does not contain any quadrangle as an induced subgraph.

The proof is complete. □

Before we proceed to the next theorem, we emphasize the following formulas and make one remark.

[1] In this paper, the multigraph is considered to be a graph with multiple edges, but no loops.

If G is a semiregular bipartite graph with n_1 and n_2 ($n_1 \geq n_2$) vertices in each colour class, then the relation

$$\begin{aligned} P_{L(G)}(\lambda) = \ & (\lambda - r_1 + 2)^{n_1 - n_2} (\lambda + 2)^{n_1 r_1 - n_1 - n_2} \cdot \\ & \prod_{i=1}^{n_2} \left((\lambda - r_1 + 2)(\lambda - r_2 + 2) - \lambda_i^2 \right) \end{aligned} \tag{2}$$

holds (compare [3], Proposition 1.2.8), where $\lambda_1, \lambda_2, \ldots, \lambda_{n_2}$ are the first n_2 largest eigenvalues of G, while each vertex of the first (resp. second) colour class has degree r_1 (resp. r_2).

Let G be a regular graph of degree s, having n vertices and m edges, then we have the relation (see [1], Theorem 2.17)

$$P_{S(G)}(\lambda) = \lambda^{m-n} P_G(\lambda^2 - s). \tag{3}$$

Remark 4.1 *Observe that both relations (2) and (3) hold even G is a regular multigraph. The proofs of the corresponding statements remain unchanged.*

Theorem 4.1 *Let G be a connected regular multigraph of degree s having n vertices and m edges ($m \geq n$). If G has the spectrum $Sp(G) = \{\lambda_1 = s, \lambda_2, \ldots, \lambda_n\}$, then the Q–spectrum of $S(G)$ is*

$$\left\{ 2^{m-n}, \frac{s + 2 \pm \sqrt{s^2 + 4(\lambda_1 + 1)}}{2}, \frac{s + 2 \pm \sqrt{s^2 + 4(\lambda_2 + 1)}}{2}, \ldots, \frac{s + 2 \pm \sqrt{s^2 + 4(\lambda_n + 1)}}{2} \right\}. \tag{4}$$

Proof Regarding Lemma 4.1, we have that $S(G)$ is a $(2, s)$–semiregular bipartite graph. In addition, it has m vertices of degree 2 and n vertices of degree s. Due to relation (3), we get that $S(G)$ has the following spectrum

$$Sp(S(G)) = \{0^{m-n}, \pm\sqrt{\lambda_1 + s}, \pm\sqrt{\lambda_2 + s}, \ldots, \pm\sqrt{\lambda_n + s}\}. \tag{5}$$

By putting $r_1 = 2, r_2 = s, n_1 = m, n_2 = n$ and $\lambda_i = \sqrt{\lambda_i + s}$ into (2), we get

$$P_{L(S(G))}(\lambda) = \lambda^{m-n} (\lambda + 2)^{m+n} \cdot \prod_{i=1}^{n} \left(\lambda(\lambda - s + 2) - (\lambda_i + s) \right).$$

Finally, due to relation (1), we get

$$Q_{S(G)}(\lambda) = (\lambda - 2)^{m-n} \cdot \prod_{i=1}^{n} ((\lambda - 2)(\lambda - s) - (\lambda_i + s)) \,.$$

This completes the proof. $\square$

By substituting $\lambda_1 = s$ into (4), we get the largest and the least Q–eigenvalue ($s + 2$ and 0, respectively).

The following simple result will be useful in sequel.

Lemma 4.2 *Let G_k ($k \geq 1$) be a multigraph (on n vertices) in which any two vertices are either non–adjacent or joined by exactly k edges. Then the eigenvalues of G_k are $k\lambda_1, k\lambda_2, \ldots, k\lambda_n$ where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of G_1.*

Proof Let A_k denote the adjacency matrix of G_k. We have

$$P_{G_k}(\lambda) = |\lambda I - A_k| = |\lambda I - kA_1| = k^n \left| \frac{\lambda}{k} I - A_1 \right| = k^n P_{G_1}\left(\frac{\lambda}{k}\right),$$

and the proof follows. $\square$

Note that any two vertices in a *complete multigraph* are joined by equal number of edges. Now, we prove the following theorem.

Theorem 4.2 *Let G be a connected regular multigraph on n vertices. The graph $S(G)$ is Q–integral if and only if G is a complete multigraph.*

Proof First, the theorem is obviously true if G is a regular multigraph on 1 or 2 vertices. Now, since no multigraph has non–integral rational eigenvalues, in view of Theorem 4.1 we get that $S(G)$ is Q–integral if and only if all the numbers $s^2 + 4(\lambda_i + 1)$ ($i = 1, 2, \ldots, n$) are the perfect squares, where λ_i ($i = 1, 2, \ldots, n$) are the eigenvalues of G, while s denotes its degree.

Clearly, $s^2 + 4(\lambda + 1)$ is a perfect square for $\lambda = -1$.

The perfect square which is nearest to s^2 is $(s \pm 1)^2$. But, the numbers s^2 and $(s \pm 1)^2$ do not have the same parity, and therefore $(s \pm 1)^2$ cannot be equal to $s^2 + 4(\lambda + 1)$ since s^2 and $s^2 + 4(\lambda + 1)$ have the same parity. The next nearest perfect square is $(s \pm 2)^2$. If we put $s^2 + 4(\lambda + 1) = (s \pm 2)^2$ we get $\lambda = \pm s$. Further, any other perfect square of the form $s^2 + 4(\lambda + 1)$ is obtained for $\lambda \notin [-s, s]$. But in this case, λ cannot be an eigenvalue of G (note, the whole spectrum lies in $[-s, s]$).

Therefore, the number $s^2 + 4(\lambda + 1)$ is a perfect square if and only if either $\lambda = -1$ or $\lambda = \pm s$. Finally, a connected regular multigraph G of degree s whose all eigenvalues belongs to $\{-s, -1, s\}$ must be a complete multigraph of degree s (see the previous lemma, if necessary).

The proof is complete. □

We finish this section with the following discussion.

In the previous theorem we determine all connected Q–integral $(2, s)$–semiregular bipartite graphs. In this way, we also found all L–integral graphs which are $(2, s)$–semiregular bipartite (see Section 1). Let us consider the integrality of graphs obtained. Let G_k be an arbitrary complete multigraph on n vertices. By virtue of Lemma 4.2, $S(G_k)$ is integral if and only if $S(G_1)$ $(= S(K_n))$ is integral. Regarding (5), we have that $S(K_n)$ $(n > 2)$ is integral whenever both $2(n-1)$ and $n-2$ are the perfect squares. Hence we have: $2(n-1) = p^2$ and $n - 2 = q^2$, for some integers p and q, i.e. $2(q^2 + 1) = p^2$. It follows that p is even and so we can write $q^2 + 1 = 2p'^2$, where $p = 2p'$. Now, we get that q is odd, and therefore $2q'^2 + 2q' + 1 = p'^2$, where $q = 2q' + 1$, or equivalently

$$q'^2 + (q' + 1)^2 = p'^2.$$

In fact, we need the Pythagorean triplets with the first two numbers successive. One can generate infinitely many such triplets by taking $q' = x^2 - y^2$ and $q' + 1 = 2xy$. This yields the well–known Pell equation $(x + y)^2 - 2x^2 = 1$, with infinitely many solutions (see [7], pp. 238–250). In this way we get another infinite series of ALQ–integral graphs (see the previous section). The first 10 Pythagorean triplets with the first two numbers successive are given in Table 2; each triplet is followed by the order of the corresponding complete graph (note that $S(K_1) = K_1$ is integral, as well).

No.	$(a, a+1, c)$	n
1.	(0, 1, 1)	3
2.	(3, 4, 5)	51
3.	(20, 21, 29)	1683
4.	(119, 120, 169)	57123
5.	(696, 697, 985)	1940451
6.	(4059, 4060, 5741)	65918163
7.	(23660, 23661, 33461)	2239277043
8.	(137903, 137904, 195025)	76069501251
9.	(803760, 803761, 1136689)	2584123765443
10.	(4684659, 4684660, 6625109)	87784138523763

Table 2

Using the values obtained, one can compute the spectra of the corresponding subdivision graphs.

5 On Q–integral $(3,4)$ and $(3,5)$–semiregular bipartite graphs

By computer search we consider the smallest graphs of Table 1 (Section 1). The results obtained are summarized in the following theorem.

Theorem 5.1 *There exists only one graph with data corresponding to the first row of Table* 1 *(G_1); there exist exactly three graphs with data corresponding to the second row (G_2, G_3 and G_4); there are no graphs having data as in the third row. In the list below, each vertex of the first colour class is represented by list of its neighbours (the vertices of the second colour class are labelled by numbers* $1, 2, \ldots, 9$*).*

$$G_1 : \left| \begin{array}{c|c|c|c|c|c} 1\,2\,3 & 4\,5\,6 & 7\,8\,9 & 1\,4\,7 & 2\,5\,8 & 3\,6\,9 \\ 1\,2\,3 & 4\,5\,6 & 7\,8\,9 & 1\,4\,7 & 2\,5\,8 & 3\,6\,9 \end{array} \right|$$

$$G_2 : \left| \begin{array}{c|c|c|c|c|c} 1\,2\,3 & 1\,2\,8 & 2\,7\,9 & 3\,6\,8 & 4\,5\,7 & 4\,6\,9 \\ 1\,2\,4 & 1\,5\,9 & 3\,5\,7 & 3\,6\,8 & 4\,6\,9 & 5\,7\,8 \end{array} \right|$$

$$G_3 : \left| \begin{array}{c|c|c|c|c|c} 1\,2\,3 & 1\,4\,7 & 2\,5\,9 & 2\,7\,8 & 3\,6\,8 & 4\,6\,9 \\ 1\,4\,7 & 1\,5\,8 & 2\,5\,9 & 3\,5\,7 & 3\,6\,8 & 4\,6\,9 \end{array} \right|$$

$$G_4 : \left| \begin{array}{c|c|c|c|c|c} 1\,2\,7 & 1\,5\,7 & 2\,4\,7 & 2\,6\,8 & 3\,5\,8 & 4\,6\,7 \\ 1\,3\,9 & 1\,5\,9 & 2\,5\,8 & 3\,4\,9 & 3\,6\,8 & 4\,6\,9 \end{array} \right|$$

Now, the only Q–integral edge–regular graphs with edge degree at most five which are not yet determined are those having any of the remaining possible spectra from Table 1. The next step is considering the graphs with edge–degree six. In fact, the connected graphs with edge–degree six are (r,s)–semiregular bipartite, where $r+s=8$ holds. With no loss of generality, we can assume that $r \leq s$.

Case $r=1$ is simple: the only solution is a star $K_{1,7}$ (recall, each complete bipartite graph is Q–integral).

Case $r=2$ leads us to the graphs obtained in the previous section.

In case $r=4$ we deal with 4–regular bipartite graphs. Having in mind that a regular graph is integral if and only if it is Q–integral (Section 1) we have that this problem is equivalent to determining the integral 4–regular bipartite graphs. Although there are some results concerning 4–regular integral graphs (see [12] and [13]) those graphs are not determined even if they are bipartite. Hence, we skip this case at this moment.

In what follows we proceed to give the possible spectra of connected Q–integral $(3,5)$–semiregular bipartite graphs. We shall need the following well known result which can be proved by considering the so–called Hoffman polynomial – see [6]: if $\lambda_1\ (=r), \lambda_2, ..., \lambda_k$ are all the distinct eigenvalues of an arbitrary regular graph G (of vertex–degree r) on n vertices, then $(r-\lambda_2)\cdots(r-\lambda_k)$ is an integer divisible by n.

Theorem 5.2 *A connected $(3,5)$–semiregular bipartite Q–integral graph has one of the Q–spectra shown in Table 3. Each row contains the number of vertices (n), the number of edges (m), the multiplicities of eigenvalues $0,1,\ldots,8$, the number of quadrangles (q) and the number of hexagons (h).*

Proof Let G be a connected $(3,5)$–semiregular bipartite graph on n vertices and m edges. Note, the least and the largest eigenvalues of G are simple and equal to 0 and 8, respectively. Let α_i denote the multiplicity of Q–eigenvalue i $(i=1,2,\ldots,7)$.

By computing we get:

$$n_1 = \frac{m}{3},\ n_2 = \frac{m}{5},\ n = n_1 + n_2 = \frac{8m}{15}, \tag{6}$$

where n_1 (resp. n_2) is the number of vertices of degree 3 (resp. 5).

By using the relations (2) and (1), we get:

$$Q_G(\lambda) = (\lambda-3)^{n_1-n_2} \cdot \prod_{i=1}^{n_2} \left((\lambda-3)(\lambda-5) - \lambda_i^2\right), \tag{7}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_{n_2}$ are the first n_2 largest eigenvalues of (the adjacency matrix of) G. Clearly, there are at least $n_1 - n_2$ Q–eigenvalues which are equal to 3, while the other Q–eigenvalues we get as the roots of the equations $\left((\lambda-3)(\lambda-5)-\lambda_i^2\right) = 0$ $(i=1,2,\ldots,n_2)$. Observe that $(\lambda-3)(\lambda-5) \geq 0$ must hold (so that the previous equations have the real roots). Therefore, each Q–eigenvalue λ of G satisfies $\lambda \in [0,3] \cup [5,8]$. In other words, we have $\alpha_4 = 0$. By using the formulas for the spectral moments T_k $(k=0,1,\ldots,6)$ (see Section 2) and having in mind the equalities (6), we arrive at the following system of Diophantine equations:

$$\begin{aligned}
&\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5 + \alpha_6 + \alpha_7 + 2 && = \frac{8}{15}m \\
&\alpha_1 + 2\alpha_2 + 3\alpha_3 + 5\alpha_5 + 6\alpha_6 + 7\alpha_7 + 8 && = 2m \\
&\alpha_1 + 2^2\alpha_2 + 3^2\alpha_3 + 5^2\alpha_5 + 6^2\alpha_6 + 7^2\alpha_7 + 8^2 && = 16m \\
&\alpha_1 + 2^3\alpha_2 + 3^3\alpha_3 + 5^3\alpha_5 + 6^3\alpha_6 + 7^3\alpha_7 + 8^3 && = 58m \\
&\alpha_1 + 2^4\alpha_2 + 3^4\alpha_3 + 5^4\alpha_5 + 6^4\alpha_6 + 7^4\alpha_7 + 8^4 && = 362m + 8q \\
&\alpha_1 + 2^5\alpha_2 + 3^5\alpha_3 + 5^5\alpha_5 + 6^5\alpha_6 + 7^5\alpha_7 + 8^5 && = 2346m + 160q \\
&\alpha_1 + 2^6\alpha_2 + 3^6\alpha_3 + 5^6\alpha_5 + 6^6\alpha_6 + 7^6\alpha_7 + 8^6 && = 15530m + 2088q + 12h
\end{aligned}$$

Solving this system, we get: $\alpha_1 = \alpha_7 = -\frac{3}{20}(4(q-45)+h)$, $\alpha_2 = \alpha_6 = \frac{1}{30}(-22q-3h+840)$, $\alpha_3 = -q-\frac{h}{4}+49$, $\alpha_5 = -\frac{q}{15}-\frac{h}{20}+7$ and $m = -7q-\frac{3h}{2}+315$. Since $n = \frac{8m}{15}$ we have that 15 divides m. Recall that line graph of a connected (r,s)–semiregular bipartite graph with m edges is a connected regular graph on m vertices. By using the result mentioned above this theorem, we find that m divides $\frac{8!}{4} = 10080$ (since $\alpha_4 = 0$, we have that 2 is not an eigenvalue of the corresponding line graph). On the other hand, since q and h are non–negative we have $m \leq 315$. Thus, $m \in \{15, 30, 45, 60, 90, 105, 120, 180, 210, 240, 315\}$. By computing the other values for every possible m, we obtain the values as in Table 3.

This completes the proof. □

n	m	0	1	2	3	4	5	6	7	8	q	h
8	15	1	0	0	4	0	2	0	0	1	30	60
16	30	1	0	5	4	0	0	5	0	1	15	120
24	45	1	3	2	9	0	3	2	3	1	30	40
32	60	1	3	7	9	0	1	7	3	1	15	100
48	90	1	6	9	14	0	2	9	6	1	15	80
56	105	1	6	14	14	0	0	14	6	1	0	140
56	105	1	9	6	19	0	5	6	9	1	30	0
64	120	1	9	11	19	0	3	11	9	1	15	60
96	180	1	15	15	29	0	5	15	15	1	15	20
112	210	1	18	17	34	0	6	17	18	1	15	0
128	240	1	21	19	39	0	7	19	21	1	15	8
168	315	1	27	28	49	0	7	28	27	1	0	0

Table 3

We consider now the smallest graphs of Table 3 (by hand, and by computer search). The results obtained are summarized in the following theorem.

Theorem 5.3 *The only graph with data corresponding to the first row of Table* 3 *is* $K_{3,5}$*; the only graph with data corresponding to the second (resp. third) row is* H_1 *(resp.* H_2*). In the list below, each vertex of the first colour class is represented by list of its neighbours (the vertices of the second colour class are labelled by numbers* $1, 2, \ldots, n_2$*).*

$$H_1 : \left| \begin{array}{c|c|c|c|c} 1\,2\,3 & 1\,3\,5 & 1\,4\,6 & 2\,4\,5 & 3\,4\,6 \\ 1\,2\,6 & 1\,4\,5 & 2\,3\,4 & 2\,5\,6 & 3\,5\,6 \end{array} \right|$$

$$H_2: \begin{array}{|c|c|c|c|c|c|c|} 1\,4\,5 & 1\,4\,9 & 1\,6\,8 & 2\,4\,9 & 2\,6\,7 & 3\,5\,7 & 3\,7\,9 \\ 1\,4\,5 & 1\,6\,8 & 2\,4\,9 & 2\,6\,7 & 2\,6\,8 & 3\,5\,8 & 3\,7\,9 \end{array}$$

We consider now the Q–spectrum given in row 6 of Table 3.

Theorem 5.4 *There does not exist a graph with data corresponding to the* 6*th row of Table* 3*.*

Proof Substituting the all Q–eigenvalues given in the 6th row into (7) we get that the corresponding distinct eigenvalues (of the adjacency matrix A) are $\pm\sqrt{15}$, $\pm\sqrt{8}$, $\pm\sqrt{3}$ and 0. Therefore, the distinct eigenvalues of matrix A^2 are 15, 8, 3 and 0.

Since $q = 0$ (and also since we are dealing with bipartite graphs), the multigraph with loops corresponding to A^2 has two components which are both graphs with loops: the first component has $n_1 = 35$ vertices and 3 loops at each vertex; the second has $n_2 = 21$ vertices and 5 loops at each vertex. So, the spectrum of the first (resp. second) component with loops excluded is contained in the following set (these eigenvalues are equal to the eigenvalues of A^2 decreased by number of loops at each vertex):

$$\{12,\ 5,\ 0,\ -3\} \quad (\text{resp. } \{10,\ 3,\ -2,\ -5\}).$$

If any graph with the corresponding Q–spectrum exists, there also exist two regular graphs (of orders 35 and 21, respectively) whose distinct eigenvalues belong to the above sets. On the other hand, the graph whose distinct eigenvalues are 10, 3, -2 and -5 does not exist (see [5], p. 250). In addition, the strongly regular graph whose distinct eigenvalues belong to $\{10,\ 3,\ -2,\ -5\}$ also does not exist (this can be easily resolved from the tables of small strongly regular graphs).

The proof is complete. □

The remaining Q–spectra of Tables 1 and 3 should be considered in forthcoming research.

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