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SEMI-ANALYTICAL FORMULAS FOR THE HERTZSPRUNG-RUSSELL DIAGRAM

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SUMMARY: The absolute visual magnitude as function of the observed color (B-V), also named Hertzsprung-Russell diagram, can be described through five equations; that when calibrated stars are available means eight constants. The developed framework allows to deduce the remaining physical parameters, mass, radius and luminosity. This new technique is applied to the first 10 pc, the first 50 pc, the Hyades and to the determination of the distance of a cluster. The case of the white dwarfs is analyzed assuming the absence of calibrated data: our equation produces a smaller χ^2 with respect to the standard color-magnitude calibration when applied to the Villanova Catalog of Spectroscopically Identified White Dwarfs. The theoretical basis of the formulas for the colors and the bolometric correction of the stars is clarified by a Taylor expansion in the temperature of the Planck distribution.

Key words. Stars: fundamental parameters – Hertzsprung-Russell (HR) and C-M – Stars: general

1. INTRODUCTION

The diagrams of absolute visual magnitude versus spectral type for the stars H-R diagram started with Hertzsprung (1905, 1911) and Rosenberg (1911). The original Russell version can be found hereafter in Russell (1914a,b,c). The common explanation is in terms of the stellar evolution, see for example chapter VII in Chandrasekhar (1967). Actually the presence of uncertainties in the stellar evolution makes the comparison between theory and observations an open field of research, (Maeder and Renzini 1984, Madore 1985, Renzini and Fusi Pecci 1988, Chiosi et al. 1992, Bedding et al. 1998). Modern application of the H-R diagram can be found in de Bruijne et al. (2001) applied to the Hyades with the parallaxes provided by Hipparcos, and in (Al-Wardat 2007) applied to the binary systems COU1289 and COU1291.

The Vogt theorem, (Vogt 1926), states that

Theorem 1 *The structure of a star is determined by its mass and its chemical composition.*

Another approach is the parametrization of physical quantities such as absolute magnitude, mass, luminosity and radius as a function of the temperature, (Cox 2000) for Morgan and Keenan classification, in the following MK, (Morgan and Keenan 1973). The temperature is not an observable quantity and, therefore, the parametrization of the observable and non observable quantities of the stars as a function of the observable color is an open problem in astronomy.

Table 1. Table of coefficients derived from the calibrated data (see Table 15.7 in (Cox 2000)) using the least square method.

	MAIN, V	GIANTS, III,	SUPERGIANTS I
$K_{\rm BV}$	-0.641 ± 0.01	-0.792 ± 0.06	-0.749 ± 0.01
$T_{\rm BV}$[K]	7360 ± 66	8527± 257	8261 ± 67
$K_{\rm BC}$	42.74 ± 0.01	44.11 ± 0.06	42.87 ± 0.01
$T_{\rm BC}$[K]	31556 ± 66	36856 ± 257	31573 ± 67
$a_{\rm LM}$	0.062 ± 0.04	0.32 ± 0.14	1.29 ± 0.32
$b_{\rm LM}$	3.43 ± 0.06	2.79 ± 0.23	2.43 ± 0.26

Table 2. Table of $a_{\rm MT}$ and $b_{\rm MT}$ with masses given in Table 3.1 of Bowers and Deeming (1984).

	MAIN,V	GIANTS,III	SUPERGIANTS,I	
			(B-V)> 0.76	(B-V) < 0.76
$a_{\rm MT}$	-7.6569	5.8958	-3.0497	4.1993
$b_{\rm MT}$	2.0102	-1.4563	-0.8491	1.0599
χ^2	28.67	3.41	20739	18.45

Conjecture 1 *The absolute visual magnitude M_V is a function, F, of the selected color*

$$M_V = F(c_1, ..., c_8, (B-V)) \quad .$$

The eight constants are different for each MK class.

In order to give an analytical expression to Conjecture 1 we first analyze the case in which we dispose presence of calibrated physical parameters for stars of the various MK spectral types, see Section 2, and then the case of absence of calibration tables, see Section 3. Different astrophysical environments such as the first 10 pc and 50 pc, the open clusters and distance determination of the open clusters are presented in Section 4. The theoretical dependence by the temperature on colors and bolometric corrections is analyzed in Section 5.

2. PRESENCE OF CALIBRATED PHYSICAL PARAMETERS

The M_V,visual magnitude, against $(B-V)$ can be found from five equations, four of which already described by (Zaninetti 2005). When the numerical value of the symbols is omitted the interested reader is referred to Zaninetti (2005). The luminosity of the star is

$$\log_{10}(\frac{L}{L_{\odot}}) = 0.4(M_{{\rm bol},\odot} - M_{\rm bol}) \quad , \qquad (1)$$

where $M_{{\rm bol},\odot}$ is the bolometric luminosity of the Sun that, according to Cox (2000), is 4.74. The equation that connects the total luminosity of a star with it's mass is

$$\log_{10}(\frac{L}{L_{\odot}}) = a_{\rm LM} + b_{\rm LM}\log_{10}(\frac{\mathcal{M}}{\mathcal{M}_{\odot}}) \quad , \qquad (2)$$

here L is the total luminosity of a star, $L_{\odot}$ is that of the Sun's, $\mathcal{M}$ and $\mathcal{M}_{\odot}$ are, respectively, the masses of the star and the Sun, $a_{\rm LM}$ and $b_{\rm LM}$ two coefficients given in Table 1 for MAIN V, GIANTS III and SUPERGIANTS I; more details can be found in Zaninetti (2005).

We recall that the tables of calibration of MK spectral types unify SUPERGIANTS Ia and SUPERGIANTS Ib into SUPERGIANTS I, see Table 15.7 in Cox (2000), and Table 3.1 in Bowers and Deeming (1984). From the theoretical side (Padmanabhan 2001) quotes $3 < b_{\rm LM} < 5$; the fit on the calibrated values gives $2.43 < b_{\rm LM} < 3.43$, see Table 1.

From an inspection of formula (2) it is possible to conclude that a logarithmic expression for the mass as function of the temperature allows us to continue with formulas easy to deal with. The following form of the mass-temperature relationship is therefore adopted

$$\log_{10}(\frac{\mathcal{M}}{\mathcal{M}_{\odot}}) = a_{\rm MT} + b_{\rm MT}\log_{10}(\frac{T}{T_{\odot}}) \quad , \qquad (3)$$

where T is the temperature of the star, $T_{\odot}$ the temperature of the Sun, $a_{\rm MT}$ and $b_{\rm MT}$ are two coefficients reported in Table 2 when the masses as function of the temperature (e.g. Table 3.1 in Bowers and Deeming (1984)) are considered. According to Cox (2000), $T_{\odot} = 5777\ K$.

Due to the fact that the masses of the SUPERGIANTS present a minimum at $(B-V) \approx 0.7$ or T$\approx$ 5700 K we have divided the analysis in two parts. Another useful formula is the bolometric correction BC

$$BC = M_{\rm bol} - M_{\rm V} = -\frac{T_{\rm BC}}{T} - 10\ \log_{10}\ T + K_{\rm BC} \quad , \qquad (4)$$

where $M_{\rm bol}$ is the absolute bolometric magnitude, M_V is the absolute visual magnitude, $T_{\rm BC}$ and $K_{\rm BC}$ are two parameters given in Table 1. The bolometric correction is always negative, but in Allen (1973) the analytical formula was erroneously reported as always positive.

The fifth equation connects the physical variable T with the observed color $(B-V)$, see, for example, Allen (1973),

$$(B-V) = K_{\rm BV} + \frac{T_{\rm BV}}{T} \quad , \tag{5}$$

where $K_{\rm BV}$ and $T_{\rm BV}$ in Table 1 are two parameters that are derived from the least square fitting procedure. Conversely, in Section 5 we shall explore a series development for $(B-V)$ as given by a Taylor series in the variable $1/T$. Inserting formulas (4) and (3) in (2) we obtain

$$M_V = -2.5\,a_{\rm LM} - 2.5\,b_{\rm LM}\,a_{\rm MT} - 2.5\,b_{\rm LM}\,b_{\rm MT}\;\log_{10} T - K_{\rm BC} + 10\;\log_{10} T + \frac{T_{\rm BC}}{T} + M_{\rm bol,\odot} \quad . \tag{6}$$

Inserting Eq. (5) in (6) the following relationship that relates M_V and $(B-V)$ in the H-R diagram is obtained

$$M_V = -2.5\,a_{\rm LM} - 2.5\,b_{\rm LM}\,a_{\rm MT} - 2.5\,b_{\rm LM}\,b_{\rm MT}\;\log_{10}(\frac{T_{\rm BV}}{(B-V)-K_{\rm BV}}) - K_{\rm BC} + 10\;\log_{10}(\frac{T_{\rm BV}}{(B-V)-K_{\rm BV}}) + \frac{T_{\rm BC}}{T_{\rm BV}}\,[(B-V)-K_{\rm BV}] + M_{\rm bol,\odot} \quad . \tag{7}$$

Up to now the parameters $a_{\rm MT}$ and $b_{\rm MT}$ are deduced from Table 3.1 in Bowers and Deeming (1984) and Table 2 reports the merit function χ^2 computed as

$$\chi^2 = \sum_{j=1}^{n} (M_V - M_V^{\rm cal})^2 \quad , \tag{8}$$

where $M_V^{\rm cal}$ represents the calibration value for the three MK classes as given in Table 15.7 of Cox (2000). From a visual inspection of the χ^2 reported in Table 2 we deduced that different coefficients of the mass-temperature relationship (3) may give better results. We therefore found by a numerical analysis the values $a_{\rm MT}$ and $b_{\rm MT}$ that minimize Eq. (8) when $(B-V)$ and $M_{\rm V}$ are given by the calibrated values of Table 15.7 in Cox (2000).

This method evaluating $a_{\rm MT}$ and $b_{\rm MT}$ is new and allows to compute them in absence of other ways to deduce the mass of a star. The absolute visual magnitude with the data of Table 3 is

$$M_{\rm V} = 31.34 - 3.365\;\ln\left(7361.0\;((B-V)+0.6412)^{-1}\right) + 4.287\,(B-V) \tag{9}$$

MAIN SEQUENCE, V when $-0.33 < (B-V) < 1.64$,

$$M_{\rm V} = -109.6 + 12.51\;\ln\left(8528.0\;((B-V)+0.7920)^{-1}\right) + 4.322\,(B-V) \tag{10}$$

GIANTS, III $0.86 < (B-V) < 1.33$,

$$M_{\rm V} = -39.74 + 3.565\;\ln\left(8261.0\;((B-V)+0.7491)^{-1}\right) + 3.822\,(B-V) \tag{11}$$

SUPERGIANTS, I when $-0.27 < (B-V) < 0.76$,

$$M_{\rm V} = -61.26 + 6.050\;\ln\left(8261.0\;((B-V)+0.7491)^{-1}\right) + 3.822\,(B-V) \tag{12}$$

SUPERGIANTS, I when $0.76 < (B-V) < 1.80$.

It is now possible to build the calibrated and theoretical H-R diagram, see Fig. 1.

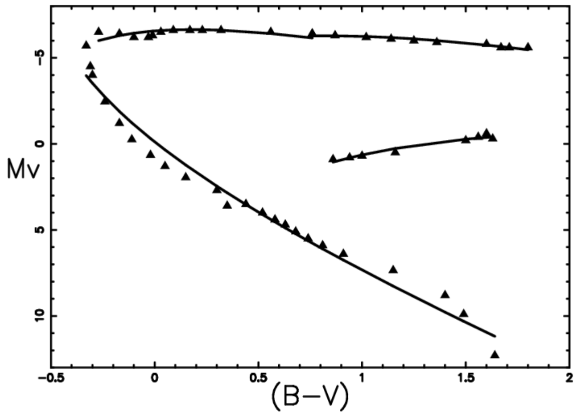

Fig. 1. *$M_{\rm V}$ against $(B-V)$ for calibrated MK stars (triangles) and theoretical relationship with $a_{\rm MT}$ and $b_{\rm MT}$ as given in Table 3.*

Table 3. Table of $a_{\rm MT}$ and $b_{\rm MT}$ when $M_V^{\rm cal}$ is given by the calibrated data.

	MAIN, V	GIANTS, III,	SUPERGIANTS, I	
			(B-V)>0.76	(B-V)<0.76
$a_{\rm MT}$	-7.76	3.41	3.73	0.20
$b_{\rm MT}$	2.06	-2.68	-0.64	0.24
χ^2	11.86	0.152	0.068	0.567

2.1. The mass vs. (B-V) relationship

It is now possible to deduce the relationship that connects the mass of the star, $\mathcal{M}$, with the variable $(B-V)$ and the constants $a_{\rm MT}$ and $b_{\rm MT}$

$$\log_{10}(\frac{\mathcal{M}}{\mathcal{M}_\odot}) = a_{MT} + b_{MT}\,\ln\left(\frac{T_{BV}}{(B-V)-K_{BV}}\right)(\ln(10))^{-1} \quad . \qquad (13)$$

With $a_{\rm MT}$ and $b_{\rm MT}$ given by Table 3 and the other coefficients taken as reported in Table 1, the following expressions for the mass are obtained:

$$\log_{10}(\frac{\mathcal{M}}{\mathcal{M}_\odot}) = 7.769 + +0.8972\,\ln\left(7360.9\,((B-V)+0.6411)^{-1}\right) \qquad (14)$$

MAIN SEQUENCE, V when $-0.33 < (B-V) < 1.64$,

$$\log_{10}(\frac{\mathcal{M}}{\mathcal{M}_\odot}) = 10.41 + -1.167\,\ln\left(8527.5\,((B-V)+0.792)^{-1}\right) \qquad (15)$$

GIANTS, III $0.86 < (B-V) < 1.33$,

$$\log_{10}(\frac{\mathcal{M}}{\mathcal{M}_\odot}) = 0.2 + +0.1276\,\ln\left(8261\,((B-V)+0.7491)^{-1}\right) \qquad (16)$$

SUPERGIANTS, I when $-0.27 < (B-V) < 0.76$,

$$\log_{10}(\frac{\mathcal{M}}{\mathcal{M}_\odot}) = 3.73 + -0.2801\,\ln\left(8261\,((B-V)+0.7491)^{-1}\right) \qquad (17)$$

SUPERGIANTS, I when $0.76 < (B-V) < 1.80$.

Fig. 2 shows the logarithm of the mass as function of $(B-V)$ for the three classes considered here (points), as well as the theoretical relationships given by Eqs. (14-17) (full lines).

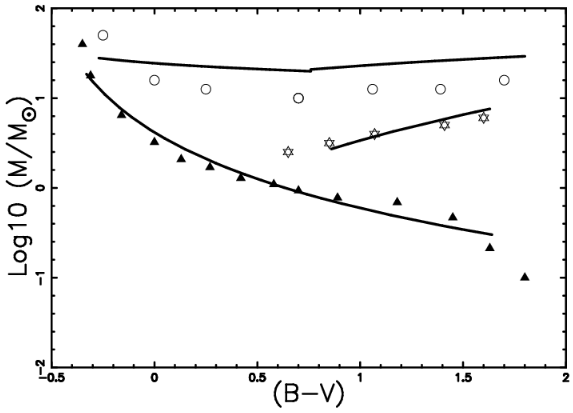

Fig. 2. $\log_{10}(\frac{\mathcal{M}}{\mathcal{M}_\odot})$ *vs.* $(B-V)$ *for calibrated MK stars : MAIN SEQUENCE, V (triangles), GIANTS, III (empty stars) and SUPERGIANTS, I (empty circles). The theoretical relationships as given by formulas (14-17) are shown with full lines.*

2.2. The radius vs. $(B-V)$ relationship

The radius of a star can be found from the Stefan-Boltzmann law, see for example formula (5.123) in Lang (1999). In our framework the radius is

$$\log_{10}(\frac{R}{R\odot}) = 1/2\,a_{LM} + 1/2\,b_{LM}\,a_{MT} + 2\,\frac{\ln(T_\odot)}{\ln(10)} + +1/2\,b_{LM}\,b_{MT}\,\ln\left(\frac{T_{BV}}{(B-V)-K_{BV}}\right)(\ln(10))^{-1} -2\,\ln\left(\frac{T_{BV}}{(B-V)-K_{BV}}\right)(\ln(10))^{-1} \quad . (18)$$

When the coefficients are as given in Table 3 and Table 1, the radius is

$$\log_{10}(\frac{R}{R\odot}) = -5.793 + 0.6729\,\ln\left(7360\,((B-V)+0.6411)^{-1}\right) \qquad (19)$$

MAIN SEQUENCE, V when $-0.33 < (B-V) < 1.64$,

$$\log_{10}(\frac{R}{R\odot}) = 22.25$$
$$-2.502 \ln\left(8527\ ((B-V)+0.7920)^{-1}\right) \quad (20)$$
GIANTS, III when
$0.86 < (B-V) < 1.33$,

$$\log_{10}(\frac{R}{R\odot}) = 8.417 -$$
$$0.7129 \ln\left(8261\ ((B-V)+0.7491)^{-1}\right) \quad (21)$$
SUPERGIANTS, I when
$-0.27 < (B-V) < 0.76$,

$$\log_{10}(\frac{R}{R\odot}) = 12.71$$
$$-1.21 \ln\left(8261\ ((B-V)+0.7491)^{-1}\right) \quad (22)$$
SUPERGIANTS, I when
$0.76 < (B-V) < 1.80$.

Fig. 3 shows the radius as function of $(B-V)$ for the three classes (points), as well as the theoretical relationships given by Eqs. (20-23) (full lines).

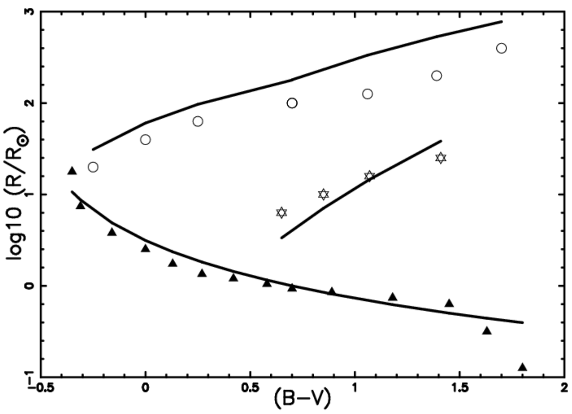

Fig. 3. $\log_{10}(\frac{R}{R\odot})$) *against* $(B-V)$ *for calibrated MK stars : MAIN SEQUENCE, V (triangles), GIANTS, III (empty stars) and SUPERGIANTS, I (empty circles). The theoretical relationships, as given by formulas (20-23), are shown with full lines.*

2.3. The luminosity vs. $(B-V)$ relationship

The luminosity of a star can be parametrized as

$$\log_{10}(\frac{L}{L\odot}) = a_{LM} + b_{LM} a_{MT}$$
$$+b_{LM}\left(b_{MT} \ln\left(\frac{T_{BV}}{(B-V)-K_{BV}}\right)\frac{1}{\ln(10)}\right) . \quad (23)$$

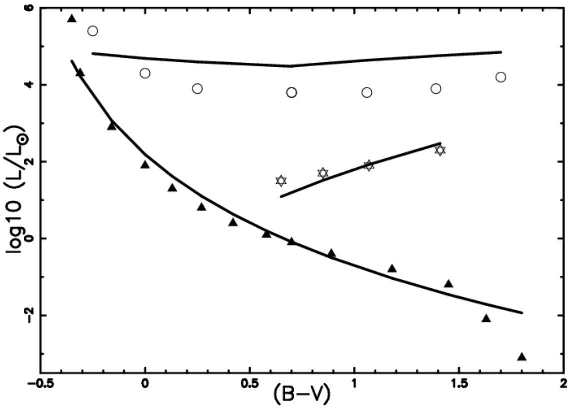

Fig. 4. $\log_{10}(\frac{L}{L\odot})$ *against* $(B-V)$ *for calibrated MK stars : MAIN SEQUENCE, V (triangles), GIANTS, III (empty stars) and SUPERGIANTS, I (empty circles). The theoretical relationships as given by formulas (25-27) are shown with full lines.*

When the coefficients are given as in Table 3 and Table 1, the luminosity is

$$\log_{10}(\frac{L}{L\odot}) = -26.63 +$$
$$+3.083 \ln\left(7360.9\ ((B-V)+0.6411)^{-1}\right) \quad (24)$$
MAIN SEQUENCE, V when
$-0.33 < (B-V) < 1.64$,

$$\log_{10}(\frac{L}{L\odot}) = 29.469 +$$
$$-3.2676 \ln\left(8527.59\ ((B-V)+0.7920)^{-1}\right) \quad (25)$$
GIANTS, III $0.86 < (B-V) < 1.33$,

$$\log_{10}(\frac{L}{L\odot}) = 1.7881 +$$
$$+0.3112 \ln\left(8261.19\ ((B-V)+0.7491)^{-1}\right) \quad (26)$$
SUPERGIANTS, I when
$-0.27 < (B-V) < 0.76$,

$$\log_{10}(\frac{L}{L\odot}) = 10.392 +$$
$$-0.682 \ln\left(8261.19\ ((B-V)+0.749)^{-1}\right)$$
SUPERGIANTS, I when
$0.76 < (B-V) < 1.80$. (27)

Fig. 4 shows the luminosity as function of $(B-V)$ for the three classes (points), as well as the theoretical relationships given by Eqs. (25-27) (full lines).

3. ABSENCE OF CALIBRATED PHYSICAL PARAMETERS

The already derived framework can be applied to a class of stars for which the calibration data of BC, $(B-V)$, $\mathcal{M}$, L versus the temperature are not available, for example to the white dwarfs. The publication of the fourth edition of the Villanova Catalog of Spectroscopically Identified White Dwarfs, makes it possible to build the H-R diagram of 568 white dwarfs that have trigonometric parallax. Once the observed absolute magnitude is derived, the merit function χ^2 is computed as

$$\chi^2 = \sum_{j=1}^{n} (M_V - M_V^{\mathrm{obs}})^2 \quad , \qquad (28)$$

where M_V^{obs} represents the observed value of the absolute magnitude, and the theoretical absolute magnitude, M_V, is given by Eq. (7). The four parameters K_{BV}, T_{BV}, T_{BC} and K_{BC} are supplied by the numerical integration of the fluxes as given by the Planck distribution. The remaining four unknown parameters a_{MT}, b_{MT}, a_{LM} and b_{LM} are supplied by minimization of Eq. (28). Table 4 reports the eight parameters that allow to build the H-R diagram.

Table 4. Table of the adopted coefficients for white dwarfs.

Coefficient	value	method
K_{BV}	-0.4693058	from the Planck law
T_{BV}[K]	6466.229	from the Planck law
K_{BC}	42.61225	from the Planck law
T_{BC}[K]	29154.75	from the Planck law
a_{LM}	0.28	minimum χ^2 on real data
b_{LM}	2.29	minimum χ^2 on real data
a_{MT}	-7.80	minimum χ^2 on real data
b_{MT}	1.61	minimum χ^2 on real data

The numerical expressions for the absolute magnitude (Eq. (7)), radius (Eq. (18)), mass (Eq. (13)) and luminosity (Eq. (23)) are:

$$M_{\mathrm{V}} = 8.199$$
$$+0.3399 \ln \left(6466.0\ ((B-V)+0.4693)^{-1}\right) +$$
$$4.509\,(B-V) \qquad (29)$$
$$\text{white dwarf}, \quad -0.25 < (B-V) < 1.88 \quad ,$$

$$\log_{10}(\frac{R}{R_\odot}) = -1.267$$
$$-0.0679 \ln \left(6466\ ((B-V)+0.4693)^{-1}\right) \qquad (30)$$
$$\text{white dwarf}, \quad -0.25 < (B-V) < 1.88 \quad ,$$

$$\log_{10}(\frac{\mathcal{M}}{\mathcal{M}_\odot}) = -7.799$$
$$+0.6992 \ln \left(6466\ ((B-V)+0.4693)^{-1}\right) \qquad (31)$$
$$\text{white dwarf}, \quad -0.25 < (B-V) < 1.88 \quad ,$$

$$\log_{10}(\frac{L}{L_\odot}) = -17.58 +$$
$$1.601 \ln \left(6466\ ((B-V)+0.4693)^{-1}\right) \qquad (32)$$
$$\text{white dwarf}, \quad -0.25 < (B-V) < 1.88 \quad .$$

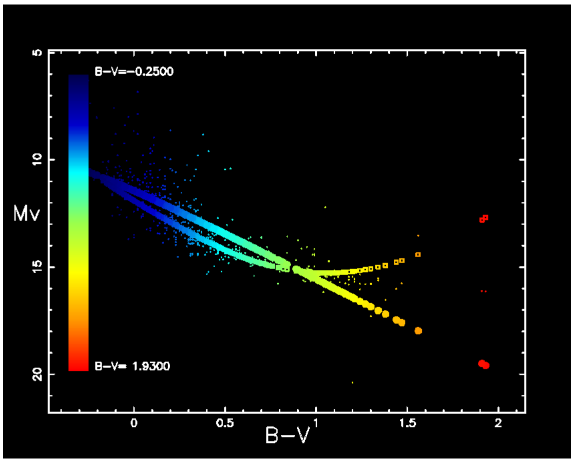

Fig. 5. *M_{V} vs. $(B-V)$ (H-R diagram) of the fourth edition of the Villanova Catalog of Spectroscopically Identified White Dwarfs. The observed stars are represented by small points, the theoretical relationship for white dwarfs by big full points and the reference relationship, given by formula (33), is represented with square points.*

Fig. 5 shows the observed absolute visual magnitude of the white dwarfs as well as the fitting curve; Table 5 reports the minimum, the average and the maximum of the three derived physical quantities.

Table 5. Derived physical parameters of the Villanova Catalog of Spectroscopically Identified White Dwarfs.

parameter	min	average	maximum
$\mathcal{M}/\mathcal{M}_\odot$	$5.28\ 10^{-3}$	$4.23\ 10^{-2}$	0.24
$R/R_\odot$	$1.07\ 10^{-2}$	$1.31\ 10^{-2}$	$1.56\ 10^{-2}$
$L/L_\odot$	$1.16\ 10^{-5}$	$2.6\ 10^{-3}$	$7.88\ 10^{-2}$

Our results can be compared with the color-magnitude relation as suggested by McCook and Sion (1999), where the color-magnitude calibration due to Dahn et al. (1982) is adopted,

$$M_V = 11.916\,((B-V)+1)^{0.44} - 0.011 \quad \text{when } (B-V) < 0.4$$
$$M_V = 11.43 + 7.25\,(B-V) - 3.42\,(B-V)^2 \quad \text{when } 0.4 < (B-V) \qquad (33)$$

The above formula (33) is shown in Fig. 5 by a line of squares, and Table 6 gives the χ^2 computed as in formula (28) for our formula (30) and for the reference formula (33). From inspection of Table 6, it is possible to conclude that our relationship represents a better fit of the data with respect to the reference formula.

Table 6. Table of χ^2 when the observed data are those of the fourth edition of the Villanova Catalog of Spectroscopically Identified White Dwarfs.

equation	χ^2
our formula (30)	710
reference formula (33)	745

The three classic white dwarfs, Procyon B, Sirius B, and 40 Eridani-B, can also be analyzed when $(B-V)$ is given (`http://www.wikipedia.org/`). The results are shown in Tables 7, 8 and 9, where we also give the data suggested in Wikipedia.

Table 7. Table of derived physical parameters of 40 Eridani-B where $(B-V)$=0.04.

parameter	here	suggested in Wikipedia
M_V	11.59	11.01
$\mathcal{M}/\mathcal{M}_\odot$	6.41 10^{-2}	0.5
$R/R_\odot$	1.23 10^{-2}	2 10^{-2}
$L/L_\odot$	3.53 10^{-3}	3.3 10^{-3}

Table 8. Table of derived physical parameters of Procyon B where $(B-V)$=0.0

parameter	here	suggested in Wikipedia
M_V	11.43	13.04
$\mathcal{M}/\mathcal{M}_\odot$	7.31 10^{-2}	0.6
$R/R_\odot$	1.21 10^{-2}	2 10^{-2}
$L/L_\odot$	4.77 10^{-3}	5.5 10^{-4}

Table 9. Table of derived physical parameters of Sirius B where $(B-V)$=-0.03.

parameter	here	suggested in Wikipedia
M_V	11.32	11.35
$\mathcal{M}/\mathcal{M}_\odot$	8.13 10^{-2}	0.98
$R/R_\odot$	1.21 10^{-2}	0.8 10^{-2}
$L/L_\odot$	6.08 10^{-3}	2.4 10^{-3}

4. APPLICATION TO THE ASTRONOMICAL ENVIRONMENT

The stars in the first 10 pc, as observed by Hipparcos (ESA 1997), belong to the MAIN V group, and Fig. 6 reports the observed stars, the calibration stars and the theoretical relationship given by Eq. (10) as a continuous line.

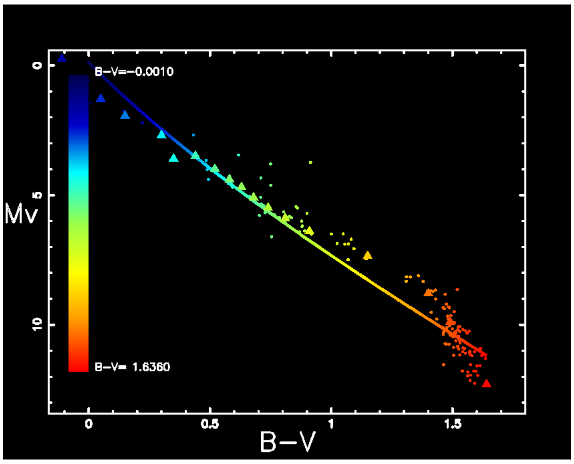

Fig. 6. *M_V against $(B-V)$ (H-R diagram) in the first 10 pc. The observed stars are represented by points, the calibrated data of MAIN V by large triangles and the theoretical relationship of MAIN V by a full line.*

The situation is different in the first 50 pc, where both the MAIN V and the GIANTS III are present, see Fig. 7, where the theoretical relationship for GIANTS III is given by Eq. (11).

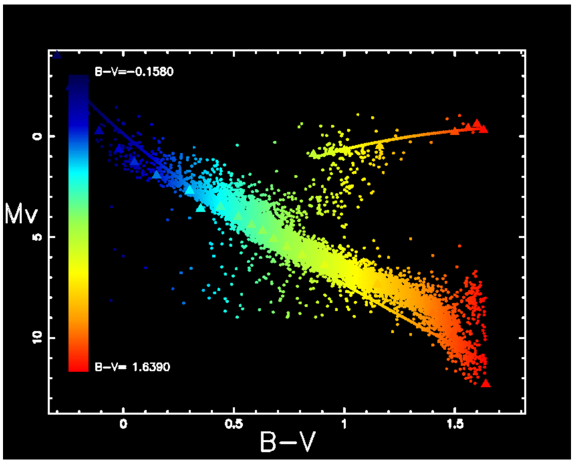

Fig. 7. *M_V against $(B-V)$ (H-R diagram) in the first 50 pc. The observed stars are represented by points, the calibrated data of MAIN V and GIANTS III by large triangles, the theoretical relationship of MAIN V and GIANTS III by full lines.*

Another astrophysical environment is that of the Hyades cluster, with $(B-V)$ and m_v as given in Stern et al. (1995) and available in the VizieR Online Data Catalog. The H-R diagram is built in absolute magnitude adopting a distance of 45 pc for the Hyades, see Fig. 8 where also the theoretical relationship of MAIN V is shown.

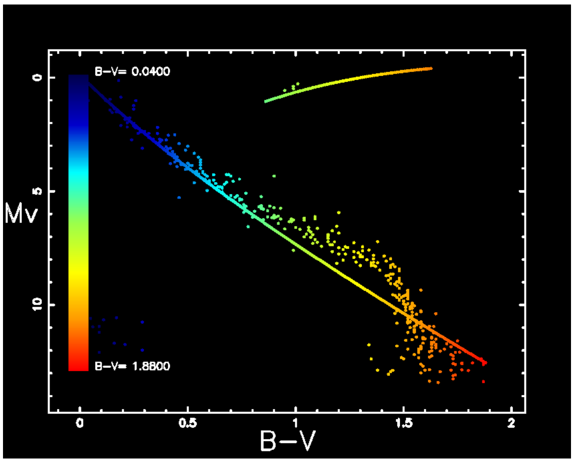

Fig. 8. *M_V against $(B-V)$ (H-R diagram) for the Hyades. The observed stars are represented by points, the theoretical relationship of MAIN V by full lines.*

Another interesting open cluster is that of the Pleiades, with the data of Micela et al. (1999) and available in the VizieR Online Data Catalog.

For the distance of the Pleiades we adopted 135 pc, according to Bouy et al. (2006); other authors suggest 116 pc as deduced from the Hipparcos data, (Mermilliod et al.1997). The H-R diagram of the Pleiades is presented in Fig. 9.

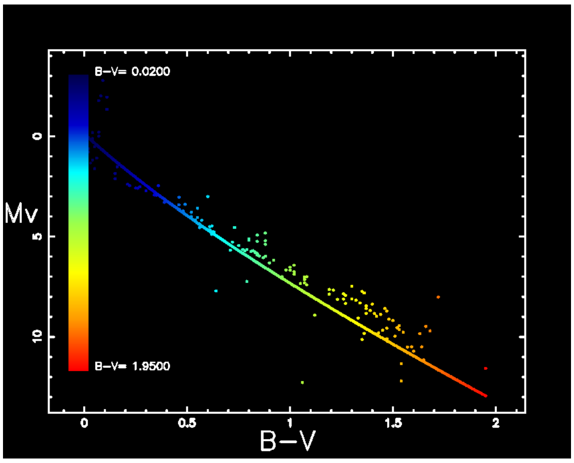

Fig. 9. *M_V against $(B-V)$ (H-R diagram) for the Pleiades. The observed stars are represented by points, the theoretical relationship of MAIN V by a full line.*

4.1. Distance determination

The distance of an open cluster can be found by the following algorithm:

(i) The absolute magnitude is computed introducing a guess value of the distance.

(ii) Only the stars belonging to MAIN V are selected. The χ^2 between observed an theoretical absolute magnitude (see Eq. (10)) is computed for different distances.

(iii) The distance of the open cluster is that connected with the value that minimizes the χ^2.

In the case of the Hyades this method gives a distance of 37.6 pc with an accuracy of 16% with respect to the guess value, or 19% with respect to 46.3 pc of Wallerstein (2000).

5. THEORETICAL RELATIONSHIPS

In order to confirm or deny the physical basis of formulas (4) and (5), we performed a Taylor-series expansion to the second order of the exact equations as given by the Planck distribution for the colors (see Section 5.1), and for the bolometric correction (see Section 5.2). A careful analysis of the numerical results applied to the Sun is reported in Section 5.3.

5.1. Colors versus Temperature

The brightness of radiation from a blackbody is

$$B_\lambda(T) = \left(\frac{2hc^2}{\lambda^5}\right) \frac{1}{\exp(\frac{hc}{\lambda kT}) - 1} \quad , \qquad (34)$$

where c is the light velocity, k is the Boltzmann constant, T is the equivalent brightness temperature and λ is the considered wavelength (see formula (13) in Planck (1901), or formula (275) in Planck (1959), or formula (1.52) in Rybicki and Lightman (1985), or formula (3.52) in Kraus (1986)).

The color-difference, (C_1 - C_2), can be expressed as

$$(C_1 - C_2) = m_1 - m_2 = K - 2.5 \log_{10} \frac{\int S_1 I_\lambda d\lambda}{\int S_2 I_\lambda d\lambda} \quad , \qquad (35)$$

where S_λ is the sensitivity function in the region specified by the index λ, K is a constant and I_λ is the energy flux reaching the Earth. We now define a sensitivity function for a pseudo-monochromatic system

$$S_\lambda = \delta(\lambda - \lambda_i) \quad i = U, B, V, R, I \quad , \qquad (36)$$

where δ denotes the Dirac delta function. In this pseudo-monochromatic color system the color-difference is

$$(C_1 - C_2) = K - 2.5 \log_{10} \frac{\lambda_2^5 \, (\exp(\frac{hc}{\lambda_2 kT}) - 1)}{\lambda_1^5 \, (\exp(\frac{hc}{\lambda_1 kT}) - 1)} \quad . \qquad (37)$$

Table 10. Johnson system.

symbol	wavelength (Å)
U	3600
B	4400
V	5500
R	7100
I	9700

The above expression for the color can be expanded in a Taylor series about the point $T = \infty$, or performing the change of variable $x = \frac{1}{T}$, about the point $x = 0$. With the expansion order 2, we have

$$(C_1 - C_2)_{\rm app} = -\frac{5}{2} \ln \left(\frac{{\lambda_2}^4}{{\lambda_1}^4} \right) \frac{1}{\ln(10)} - \frac{5}{4} \frac{hc\,(\lambda_1 - \lambda_2)}{\lambda_2 \, \lambda_1 \, k \ln(10) \, T} - \frac{5}{48} \frac{h^2 c^2 \left({\lambda_1}^2 - {\lambda_2}^2 \right)}{{\lambda_2}^2 {\lambda_1}^2 k^2 \ln(10) \, T^2} \quad , (38)$$

where the index app means approximated. We now continue by inserting the value of the physical constants as given by CODATA (Mohr and Taylor 2005) and wavelength of the color as given in Table 15.6 of Cox (2000) and reported in Table 10. The wavelength of U, B and V are exactly the same as those of the multicolor photometric system defined by Johnson (1966); conversely, R(7000 $\AA$) and I(9000 $\AA$) as given by Johnson (1966) are slightly different from the values used here. We now parameterize the color as

$$(C_1 - C_2)_{\rm app} = a + \frac{b}{T} + \frac{d}{T^2} \quad . \qquad (39)$$

Another important step is the calibration of the color on the maximum temperature $T_{\rm cal}$ of the reference tables. For example for MAIN SEQUENCE V at $T_{\rm cal} = 42000$ (see Table 15.7 in Cox (2000)), $(B - V) = -0.3$ and, therefore, a constant should be added to formula (38) in order to obtain such a value. With these recipes we obtain, for example

$$(B - V) = -0.4243 + \frac{3543}{T} + \frac{17480000}{T^2} \qquad (40)$$

MAIN SEQUENCE, V

$-0.33 < (B - V) < 1.64.$

The basic parameters b and d for the four colors here considered are reported in Table 12. The parameter a, when WHITE DWARF, MAIN SEQUENCE, V, GIANTS, III and SUPERGIANTS, I are considered, is reported in Table 11, likewise the Table 12 shows the coefficients b and d that are common to the classes of stars considered here. The WHITE DWARF calibration is made on the values of $(U - B)$ and $(B - V)$ for Sirius B, (`http://www.wikipedia.org/`).

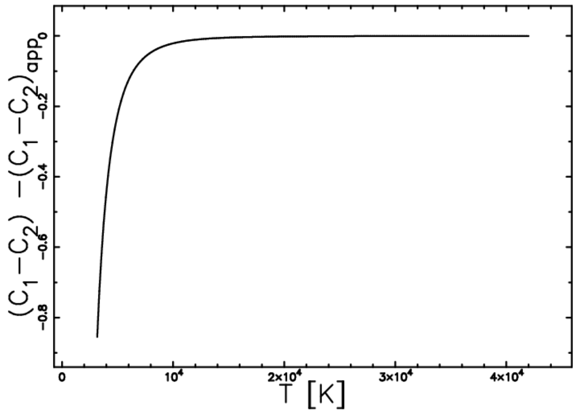

Fig. 10. *Difference between $(B - V)$, the exact value from the Planck distribution, and $(B - V)_{app}$ approximate value as deduced from the Taylor expansion for MAIN SEQUENCE, V.*

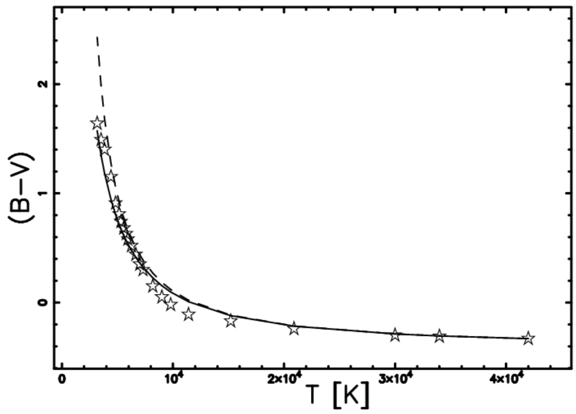

Fig. 11. *Exact $(B - V)$ as deduced from the Planck distribution, or Eq. (37), shown by a full line. Approximate $(B - V)$ as deduced from the Taylor expansion, or Eq. (38), traced with a dashed line. The calibrated data for MAIN SEQUENCE V are extracted from Table 15.7 in Cox (2000) and are represented by empty stars.*

The Taylor expansion agrees very well with the original function and Fig. 10 shows the difference between the exact function as given by the ratio of two exponential and the Taylor expansion in the $(B - V)$ case.

In order to establish a range of reliability of the polynomial expansion we solve the nonlinear equation

$$(C_1 - C_2) - (C_1 - C_2)_{\rm app} = f(T) = -0.4 \quad , \quad (41)$$

for T.

Table 11. Coefficient a.

	(B-V)	(U-B)	(V-R)	(R-I)
WHITE DWARF, $T_{\rm cal}$[K]=25200	- 0.1981	- 1.234		
MAIN SEQUENCE, V, $T_{\rm cal}$[K]=42000	- 0.4243	- 1.297	- 0.233	- 0.395
GIANTS III, $T_{\rm cal}$[K]=5050	- 0.5271	- 1.156	- 0.4294	- 0.4417
SUPERGIANTS I, $T_{\rm cal}$[K]=32000	- 0.3978	- 1.276	- 0.2621	- 0.420

The solutions of the above nonlinear equation are reported in Table 13 for the four colors considered here. For the critical difference we have chosen the value -0.4 that approximately corresponds to 1/10 of the range of existence in $(B-V)$.

Fig. 11 reports the exact and the approximate value of $(B-V)$ as well as the calibrated data.

Table 12. Coefficients b and d.

	(B-V)	(U-B)	(V-R)	(R-I)
b	3543	3936	3201	2944
d	17480000	23880000	12380000	8636000

Table 13. Range of existence of the Taylor expansion for MAIN SEQUENCE, V

	(B-V)	(U-B)	(V-R)	(R-I)
$T_{\rm min}$ [K]	4137	4927	3413	2755
$T_{\rm max}$ [K]	42000	42000	42000	42000

5.2. Bolometric Correction versus Temperature

The bolometric correction BC, defined as always negative, is

$$BC = M_{\rm bol} - M_{\rm V} \quad , \qquad (42)$$

where $M_{\rm bol}$ is the absolute bolometric magnitude and M_V is the absolute visual magnitude. It can be expressed as

$$BC = \frac{5}{2}\frac{\ln\left(15(\frac{hc}{kT\pi})^4(\frac{1}{\lambda_V})^5\frac{1}{\exp(\frac{hc}{kT\lambda_V})-1}\right)}{\ln(10)} + K_{BC} \quad , \qquad (43)$$

where λ_V is the visual wavelength and K_{BC} a constant. We now expand into a Taylor series about the point $T=\infty$

$$BC_{\rm app} = -\frac{15}{2}\frac{\ln(T)}{\ln(10)} - \frac{5}{4}\frac{hc}{k\lambda_V\,\ln(10)\,T} - \frac{5}{48}\frac{h^2c^2}{k^2{\lambda_V}^2\ln(10)\,T^2} + K_{BC} \quad . \qquad (44)$$

The constant K_{BC} can be found by the following way. The maximum of $BC_{\rm app}$ is at $T_{\rm max}$, where the index max stands for

$$T_{\rm max} = \frac{1}{6}\frac{\left(\frac{\sqrt{5}}{2}+\frac{1}{2}\right)ch}{k\lambda_V} \quad . \qquad (45)$$

Given the fact that the observed maximum in the BC is -0.09 at 7300 K in the case of MAIN SEQUENCE V we easily compute K_{BC}, and the following approximate result is obtained

$$BC_{\rm app} = 31.41 - 3.257\,\ln(T) - \frac{14200}{T} - \frac{3.096\;10^7}{T^2} \quad . \qquad (46)$$

Fig. 12 reports the exact and the approximate value of BC as well as the calibrated data.

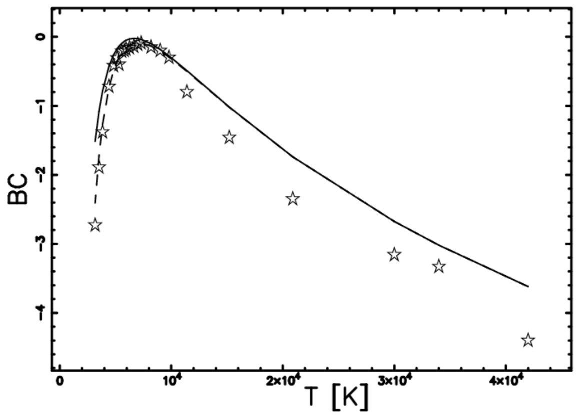

Fig. 12. *Exact BC as deduced from the Planck distribution, Eq. (43), shown by a full line. Approximate BC as deduced from the Taylor expansion, Eq. (46), shown by a dashed line. The calibrated data for MAIN SEQUENCE V are extracted from Table 15.7 in Cox (2000) and are represented by empty stars.*

The Taylor expansion agrees very well with the original function and Fig. 13 shows the difference between exact function, Eq. (43), and the Taylor expansion, Eq. (46).

In order to establish the range of reliability of the polynomial expansion, we solve for T the nonlinear equation

$$BC - BC_{\rm app} = F(T) = -0.4 \quad . \qquad (47)$$

The solution of this nonlinear equation allows to state that the bolometric correction as derived from a Taylor expansion for MAIN SEQUENCE, V is reliable in the range $4074\;K < T < 42000\;K$.

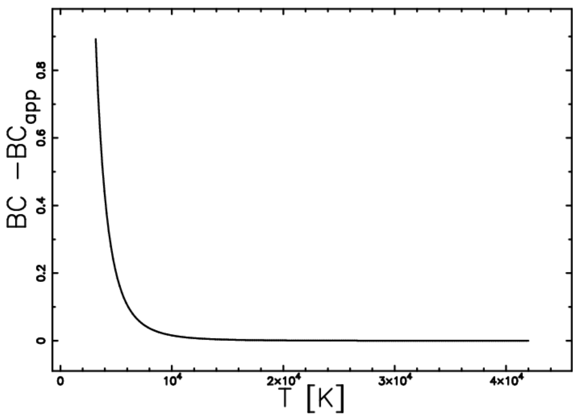

Fig. 13. *Difference between $(B-V)$, the exact value obtained from the Planck distribution, and $(B-V)_{app}$, the approximate value deduced from the Taylor expansion for MAIN SEQUENCE, V.*

5.3. The Sun as a black body radiator

The framework previously derived allows to apply our formulas to one specific star of spectral type G2V with T=5777 K namely, the Sun. In order to make a comparison we give in Table 14 the various values of $(B-V)$ as reported by different methods. In Table 15 we give the value of the infrared color (R-I). From a careful examination of the two tables we conclude that our model works more properly in the far-infrared window than in the optical one.

Table 14. (B-V) of the Sun, T=5777 K.

meaning	(B-V)
calibration, Cox (2000)	0.65
here, Taylor expansion	0.711
here, Planck formula	0.57
least square method, Zaninetti (2005)	0.633
Allen (1973)	0.66
Sekiguchi and Fukugita (2000)	0.627
Johnson (1966)	0.63

Table 15. (R-I) of the Sun, T=5777 K.

meaning	(R-I)
calibration, Cox (2000)	0.34
here, Taylor expansion	0.37
here, Planck formula	0.34

6. CONCLUSIONS

New formulas. A new analytical approach based on five basic equations allows to connect the color $(B-V)$ of the stars with the absolute visual magnitude, the mass, the radius and the luminosity.The suggested method is based on eight parameters that can be precisely derived from the calibration tables; this is the case of MAIN V, GIANTS III and SUPERGIANTS I. In the absence of calibration tables, the eight parameters can be derived mixing four theoretical parameters extracted from Planck distribution with four parameters that can be found minimizing the χ^2 connected with the observed visual magnitude ; this is the case of white dwarfs. In the case of white dwarfs the mass-luminosity relationship, see Table 4, is

$$\log_{10}(\frac{L}{L_{\odot}}) = 0.28 + 2.29 \log_{10}(\frac{\mathcal{M}}{\mathcal{M}_{\odot}}) \quad , \qquad (48)$$

with $0.005\mathcal{M}_{\odot} < \mathcal{M} < 0.24\mathcal{M}_{\odot}$ for white dwarfs.

Applications. The application of the new formulas to the open clusters such as Hyades and Pleiades allows to speak of universal laws for the star's main parameters. In the absence of accurate methods to deduce the distance of an open cluster, an approximate evaluation can be carried out.

Theoretical bases. The reliability of an expansion to the second order of the colors and bolometric correction for stars as derived from the Planck distribution is carefully explored and the range of existence in temperature of the expansion is determined.

Inverse function. In this paper we have chosen a simple hyperbola for $(B-V)$ as function of the temperature, formula (5). This function can be easily inverted in order to obtain T as function of $(B-V)$ (MAIN SEQUENCE, V)

$$T = \frac{7360}{(B-V) + 0.641} \, K \qquad (49)$$
$$\text{MAIN SEQUENCE, V}$$
$$\text{when } 4137 < T[K] < 42000$$
$$\text{or when } -0.33 < (B-V) < 1.45 \quad .$$

When a more complex relationship is chosen, for example a two degree polynomial expansion in $\frac{1}{T}$ as given by formula (38), the inverse formula that gives T as function of $(B-V)$ (MAIN SEQUENCE, V) is more complicated,

$$T = 5000 \times$$
$$\frac{0.35\ 10^8 + \sqrt{4.21\,10^{15} + 6.96\,10^{15}(B-V)}}{10^8(B-V) + 0.42 10^8} \, K \qquad (50)$$
$$\text{MAIN SEQUENCE, V}$$
$$\text{when } 4137 < T[K] < 42000$$
$$\text{or when } -0.33 < (B-V) < 1.45 \quad .$$

The mathematical treatment that allows to deduce the coefficients of the series reversion can be found in Morse and Feshbach (1953), Dwight (1961), Abramowitz and Stegun (1965).

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СЕМИ-АНАЛИТИЧКЕ ФОРМУЛЕ ЗА H-R ДИЈАГРАМ

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Оригинални научни рад

Комбинацијом пет релација између основних физичких карактеристика звезда изводи се зависност апсолутне магнитуде M_V од посматране боје $(B - V)$; релација садржи осам коефицијената чије вредности зависе од класе сјаја. Такође, изводе се формуле за масу, радијус и луминозност звезде у функцији боје. Експлицитан облик ове четири релације даје се за три класе сјаја (I, III, V) H-R дијаграма. Тестирање добијене зависности $M_V(B - V)$ на два узорка звезда Hipparcos каталога – до 10 pc (класа сјаја V) и до 50 pc од Сунца (класе V, III), као и примена релације у одређивању даљина до два блиска расејана звездана јата (Хијаде и Плејаде) даје добре резултате. У раду се предлаже и метод за одређивање ових зависности код звезда за које не постоје калибрисане релације за боју, болометријску поправку, масу и луминозност у зависности од температуре. Резултати се тестирају на посматраном узорку (Villanova Catalog) блиских белих патуљака.